

On the fundamental length of quantum geometry and the black hole entropy

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The geometric operators of area, volume, and length, depend on a fundamental length ℓ of quantum geometry which is a priori arbitrary rather than equal to the Planck length ℓ_P . The fundamental length ℓ and the Immirzi parameter γ determine each other. With any ℓ the entropy formula is rendered most naturally in units of the length gap $\sqrt{\sqrt{3}/2}(\sqrt{\gamma}\ell)$.

Independently of the choice of ℓ , the black hole entropy derived from quantum geometry in the limit of classical geometry is completely consistent with the Bekenstein-Hawking form.

The extremal limit of 1-puncture states of the quantum surface geometry corresponds rather to an extremal string than to a classical horizon.

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In [1] the black hole entropy for the quantum geometry of the exterior horizon of a black hole has been calculated from first principles. The area is formally proportional to the (real modulus of) the Immirzi [2] parameter γ , parametrizing certain affine transformations of the underlying phase space which maintain the symplectic structure. The result of [1] agrees for one particular value

$$\gamma = \gamma_0 := \frac{\ln 2}{\pi\sqrt{3}} \quad (1)$$

with the Bekenstein-Hawking (BH) formula. In [3] certain extremal states of the quantum horizon geometry were examined. It was shown that, under the assumption that the (semi-)classical limit of these states existed and saturated a Bogomol'nyi type bound as for a classical Kerr black hole, these states would yield $\gamma \neq \gamma_0$ providing an apparent contradiction to the result of [1]. However, as is pointed out below, these states do not admit a semiclassical limit of quantum geometry. Hence one might think that γ might scale and, in particular, take a different value for quantum and classical geometry. But, apart from contradicting Occam's razor, such a scaling of γ would explicitly break the very symplectic structure on which the quantization of geometry is based. Indeed the results below show that the argument of [3] is independent of either γ or the choice of the fundamental length ℓ . It only depends on a combination of the two.

Hence here we will argue that the result of [1] has to be correct a fortiori, if and only if the state space is considered in the classical limit of quantum geometry.

It should also be noted here that string theory calculations reproduce the BH formula, because like the BH they are performed in the same semiclassical regime of quantum fields in a certain background geometry.

The classical limit of quantum geometry could be characterized by discrete spectra of the geometric operators like area, volume [4,5] and length [6] becoming dense on a sufficiently large state space. (The area spectrum has been proven rigorously to become dense in [7]. It appears a reasonable conjecture that the volume spectrum [8] does so too, since both operators are quite analogous in the sense that they are constructed as the vanishing regulator limit of the square root of appropriately regulated expressions quadratic and cubic respectively in the momenta of the holonomy variables. The length operator however is quite different, since it requires actually a tubular regularization of the curve and the defining regulated expression involves commutators of the holonomy variables and the volume operator.)

Quantum geometry is now just on the edge to make for the first time contact with its classical limit, the example under investigation being the spatial black hole horizon geometry and the entropy of its state space. Entropy is distinguished by the fact that it is physically dimensionless; it requires just counting. Hence it is intrinsically a scale invariant entity. This is of advantage when the classical limit is investigated. Once the classical limit is understood completely, quantum geometry has achieved the first fundamental derivation of the black hole entropy from an underlying microscopic theory. However the classical limit is quite tricky: If the state space of quantum surface geometries is restricted to particular spin networks the classical limit may fail to exist.

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In the present approach we start with arbitrary fundamental length ℓ and arbitrary γ . After resuming the results for entropy and area, we try to understand better the origin of the distinguished value obtained in [1].

The exterior horizon H of a classical black hole geometry is an inner null boundary of its exterior space-time M . As in [1] here we consider pullbacks of the phase space from M to a spatial (say Cauchy-like) slice Σ which hits H transversally on $S := \partial\Sigma \cong S^2$. Let us consider canonical the pulled-back real phase space variables $({}^\gamma A, {}^\gamma E) := (\Gamma - \gamma K, \frac{1}{\gamma} E)$ on Σ , where Γ and K are spin connection and curvature 1-forms and E is a 2-form dual to the triad on Σ . Let $\gamma\delta \equiv (\delta^\gamma A, \delta^\gamma E)$ denote a tangent vector on phase space at $({}^\gamma A, {}^\gamma E)$. Here ${}^\gamma A$ is dimensionless and ${}^\gamma E$ has physical dimension (length)². The symplectic structure is then given as by

$$\begin{aligned}\Omega|_{({}^\gamma A, {}^\gamma E)}(\gamma\delta, \gamma\delta') &:= \frac{1}{\ell^2} \int_\Sigma \text{Tr} [\delta^\gamma E \wedge \delta^\gamma A' - \delta^\gamma E' \wedge \delta^\gamma A] \\ &\quad - \frac{k}{2\pi} \int_S \text{Tr} [\delta^\gamma A \wedge \delta^\gamma A']\end{aligned}\tag{2}$$

on the real phase, where $k := \frac{A_S}{\gamma\ell^2}$ and A_S is the area of S . It has been shown in [1] how (2) is related to an action with $U(1)$ boundary Chern-Simons (CS) term representing a natural choice of boundary conditions and such that the Einstein equations are reproduced locally. Here the only difference to the symplectic structure used in [1] is the normalization. Their normalization would formally correspond to $\ell = 8\pi\ell_P$ while in our approach ℓ is just *some* microscopic fundamental length. (Note that the classically convenient factor 8π of two times the unit sphere area embedded in flat space for quantum geometry makes no sense *a priori*.) It eventually becomes fixed only after comparison with the semiclassical BH-entropy formula has been made. Therefore, here the prefactor of the classical action before quantization is left *a priori* arbitrary rather than fixing it bona fide to the Newton coupling. As long as we do not compare the microscopic quantum geometry to its classical limit in the presence of some further (matter) fields, there appears to be no necessity to fix it in any particular manner. If the quantum theory of geometry is obtained by quantizing a classical action we can not expect that the classical couplings are *a priori* the right ones, while in fact they have to be chosen *a posteriori* such that they yield the correct values in the physical classical limit. There is no obstruction that finally a classical limit of quantum geometry recovers indeed all the mathematical structure of the formal classical action put in before the quantization. But the numerical values of the classical coupling constants are only determined by experiments in the classical limit of geometry rather than being predetermined by quantum geometry. The asymptotic regime of observation within a classical geometry is the only one which is directly accessible to us while the regime of quantum geometry is physically quite different. Indeed even string theory (while still in the realm of classical background geometry) predicts already a different coupling at the scale ℓ_s where residue dilatonic fields from extra dimensions combine to couple to gravity. Therefore the microscopical coupling constants of quantum geometry should only be determined in the classical limit by consistency with the observed ones. With $\ell \ll \ell_s$, and ℓ_s close to ℓ_P within few orders of magnitude only, it appears rather implausible that ℓ could equal ℓ_P .

Consider now a finite set P of discrete transversal punctures (labeled $p = 1, \dots, N$) of S by edges from a *gauge invariant* spin network. The degrees of freedom on its surface are initially given by a choice of a $SU(2)$ -representation of spin j_p at each puncture p , induced by an intersecting edge of the $SU(2)$ cylindrical state of the same spin j_p . Identifying the transversal components of the $SU(2)$ connection in the representation of spin j along the intersecting edge with an $U(1)$ connection, this then induces a $U(1)$ Chern-Simons state on the surface. The trivial $U(1)$ -representation has then a multiplicity $2j + 1$ which also is the dimension of the corresponding Hilbert space. So on the S^2 surface the $SU(2)$ gauge symmetry is broken to $U(1)$. In the $SU(2)$ spin network representation with spins $\{j_p\}_{p \in P}$ at the punctures P on S , the area operator is diagonal with eigenvalue

$$A_S^P := \gamma\ell^2 \sum_{p \in P} \sqrt{j_p(j_p + 1)}.\tag{3}$$

Like in [1] counting of states fixes the entropy as

$$S^P = \frac{\gamma_0}{\gamma\ell^2} 2\pi A_S^P = \ln 2 \frac{2}{\sqrt{3}} \sum_{p \in P} \sqrt{j_p(j_p + 1)},\tag{4}$$

where the γ_0 is given by (1). Now in the classical limit the result should agree with the BH formula. Hence in this limit

$$\gamma\ell^2 \stackrel{!}{=} 8\pi\gamma_0\ell_P^2,\tag{5}$$

which only fixes the combination $\sqrt{\gamma}\ell$.

The formula (4) takes a more elegant form when rendered in terms of the length gap

$$\lambda := \sqrt{\sqrt{3}/2}(\sqrt{\gamma}\ell), \quad (6)$$

the lowest eigenvalue of the length operator. In fact it would be assumed here for any curve segment in S intersecting the divalent vertex given by the puncture p on S^2 of any edge with spin $j_p = 1/2$. Let

$$a := 2\pi\lambda^2 \quad (7)$$

be a half the area of a sphere of radius λ . Then the entropy formula simply reads

$$e^{S^P} = 2^{A_S^P/a}. \quad (8)$$

So (6) and (7) render the microscopical entropy formula the simple form (8), while (5) guarantees agreement with the BH formula in the classical limit. E.g. for a state with N punctures of spin $\frac{1}{2}$ (a state from bits) it holds $A_S^P/a = 4N$ and (8) simply reads $e^{S^P} = 2^{4N}$.

In order to illustrate the difficulty with the classical limit, let us now consider the space Q of surface states given by a finite number of punctures with all edges of same spin j and restricted to meet S transversally only. The spectrum restricted to these states is

$$\text{Spec} := \{\gamma\ell^2 N \sqrt{j(j+1)}, N, j \in \mathbb{N}_0\}. \quad (9)$$

Let us now consider the particular subspace Q_1 given by the 1-puncture states considered in [3]. Here $A_S := \gamma\ell^2 \sqrt{j(j+1)}$ where j is restricted to integer values by gauge invariance since S is the closed inner boundary of Σ . In the limit $j \rightarrow \infty$, the bound

$$j < \frac{A_S}{\gamma\ell^2}, \quad (10)$$

becomes saturated in leading order, i.e.

$$\frac{A_S}{\gamma\ell^2 j} - 1 \rightarrow +0. \quad (11)$$

Under the assumption that $j \rightarrow \infty$ was a classical limit for the surface geometry, and $J := \hbar j$ corresponded here to a classical angular momentum one might postulate that in the classical limit (10) corresponds to an asymptotic saturation of

$$A_S/(8\pi G) > J/c^3, \quad (12)$$

where G is the Newton constant and c is the velocity of light. (A bound like (12) would e.g. be satisfied for the spatial horizon area A_S and angular momentum of a Kerr black hole.) (11) implied then in particular that

$$\gamma = 8\pi(\ell_P/\ell)^2. \quad (13)$$

Hence for *any* possible choice of ℓ there would be a contradiction between (13) and (5) unless $\gamma_0 = 1$ which then would seem to be in contradiction to the calculated value of [1].

However, the 1-puncture configurations considered here are a very particular subspace Q_1 of the full configuration space only whence the entropy with respect to Q_1 has as pointed out in [3] indeed to be less than the entropy of [1].

Moreover, these highly degenerate Q_1 configurations never yield a classical limit for $j \rightarrow \infty$. The spectrum Spec_1 of the area operator on Q_1 has differences between neighboring eigenvalues which asymptotically become $\gamma\ell^2$ for $j \rightarrow \infty$. Hence Spec_1 does not become dense in the limit $j \rightarrow \infty$. (Note that this is unlike the length spectrum [6] which becomes dense for $j \rightarrow \infty$ even when restricted to Q_1 since the distance of neighboring eigenvalues there is proportional to $1/\sqrt{j}$, for curve segments in S containing the puncture.)

In fact, since all the area is concentrated in a single point on S , the limit $j \rightarrow \infty$ corresponds rather to an infinitely extended string than to a horizon surface. The corresponding limit state appears as the quantum analogue of a string hair leaving the extremely degenerate horizon at the vortex. Geometrically, one could indeed say that it is just a string hair without any horizon. Abelian Nielsen-Olesen string hairs emerging from a vortex were proven to exist

and have been examined in quite some detail within the Abelian Higgs model [9]- [12]. There the $U(1)$ symmetry is broken everywhere but at the vortex by a Higgs potential. For the surface states considered here the local $U(1)$ gauge symmetry , which was obtained by particular boundary conditions on S , on Q_1 spin networks is trivially also global. However, the topological boundary $S = \partial\Sigma$ is nowhere represented geometrically but at the puncture, where the transverse intersecting edge plays the analogue of an axis of rotation. Due to the asymptotic saturation of (12), it is tempting to conjecture that the $j \rightarrow \infty$ limit of the quantum string hair corresponds actually to a BPS state.

From above examples it is obvious that the classical limit of increasingly dense spectrum requires to admit a variable number of punctures and a mixture of different spins j_p (this was used in the density proof for the area operator given in [7]). Besides, at least in principle one has to take into account also states with edges on the surface S and trivalent punctures.

Once we are able to perform the classical limit with a sufficiently large state space then the counting of the number of surface states on S will yield the entropy independent of γ and independent of ℓ . When fitted to the BH formula in the classical limit, the fundamental length γ and the Immirzi parameter γ are no longer independent since in this limit the unknown the interaction between quantum geometry and quantum matter has to approach the known semiclassical coupling of classical geometry to quantum matter a fortiori. In fact $\gamma\ell^2$ enters the matter fields and their momenta with a power determined by the spin of the matter field [13]. One might hope that TQFT (see e.g. [14]) may once be able to provide a model for the quantization of both geometry and matter. The former quantization introduces necessarily the fundamental length ℓ , while the quantization of matter has to yield usual field quantization which necessarily invokes the Planck constant \hbar . In fact it is only then that the Newton coupling related to $G/c^3 = \ell_P^2/\hbar$ becomes to play a nontrivial role. However to fix the value of both ℓ and γ requires something more, namely a more detailed understanding *how* quantum geometry converges towards its (semi-)classical limit.

Let us conclude emphasizing again, that at present all results from quantum geometry are perfectly consistent with the BH formula for the black entropy. With (8) the entropy formula takes also microscopically a simple form. While the classical limit fixes the relation between γ and the fundamental length ℓ , on the microscopic level quantum geometry and quantum matter may still interact with a coupling different from the one of the classical limit.

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- [1] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Quantum Geometry and Black Hole Entropy, *Phys. Rev. Lett.* **80**, No. 5, 904-907 (1998).
- [2] G. Immirzi, Quantum Gravity and Regge Calculus, *Nucl. Phys. Proc. Suppl.* **57** 65-72 (1997).
- [3] K. Krasnov, Quanta of geometry and rotating black holes, gr-qc/9902015.
- [4] C. Rovelli and L. Smolin, Discreteness of area and volume in quantum gravity, *Nucl. Phys.* **B442**, 593 (1995); Erratum: *Nucl. Phys.* **B456**, 734 (1995).
- [5] R. De Pietri and C. Rovelli, Geometry Eigenvalues and Scalar Product from Recoupling Theory in Loop Quantum Gravity, *Phys. Rev.* **D54**, 2664 (1996).
- [6] T. Thiemann, A length operator for canonical quantum gravity, gr-qc/9606092.
- [7] A. Ashtekar and J. Lewandowski, Quantum Theory of Geometry I: Area operators, *Class. Quant. Grav.* **14**, 55 (1997).
- [8] A. Ashtekar and J. Lewandowski, Quantum Theory of Geometry II: Volume Operators, *Adv. Theor. Math. Phys.* **1**, No. 2 (1998).
- [9] A. Achúcarro, R. Gregory, and K. Kuijken, *Phys. Rev. D* **52**, 5729-5742 (1995).
- [10] A. Achúcarro and R. Gregory, *Phys. Rev. Lett.* **79**, 1972-1975 (1997).
- [11] A. Chamblin, J.M.A. Ashbourn-Chamberlin, R. Emparan, and A. Sornborger, *Phys. Rev. D* **58**, 124014 (1998).
- [12] F. Bonjour, R. Emparan, R. Gregory, Vortices and extreme black holes: the question of flux expulsion, gr-qc/9810061.
- [13] L.J. Garay, G.A. Mena Marugan, *Class.Quant.Grav.* **15** (1998) 3763-3775
- [14] J. Baez, Higher-Dimensional Algebra and Planck-Scale Physics, gr-qc/9902017.